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Minimal Projections onto Two Dimensional Subspaces of $l_\infty^{(4)}$

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Let Y be a two-dimensional subspace of $l_\infty^{(4)}$. A formula for a minimal projection from $l_\infty^{(4)}$ onto Y will be presented. Also, a complete characterization of the unicity of this minimal projection will be given

1. INTRODUCTION

Let X be a normed space and let Y be a linear subspace of X. A bounded linear operator $P: X \to Y$ is called a projection if $Py = y$ for any $y \in Y$. Denote by $\mathscr{P}(X, Y)$ the set of all projections from X onto Y. A projection P_o is called minimal if

$$\|P_o\| = \lambda(Y, X) = \inf\{\|P\| : P \in \mathscr{P}(X, Y)\}. \tag{1.1}$$

The significance of this notion can be illustrated by the following well known inequality

$$\|x - Px\| \leqslant \|Id - P\| \cdot \operatorname{dist}(x, Y) \leqslant (1 + \|P\|) \cdot \operatorname{dist}(x, Y)$$

for every $x \in X$ and $P \in \mathscr{P}(X, Y)$. For more complete information about this subject the reader is referred to [BarP, BarL, BlCh, ChaM, CheL, CheM, CheP, Fr, Ki, LE1, LE2, LE3, Od, OdL, Ro, Wo, Wu]. In general, there are two principal methods for seeking minimal projections. The first of them is based on Rudin's theorem (see e.g. [Ru, Chap. 5; CheL; or Wo. p. 118]). In [Wo, Chap. III B] some applications of Rudin's theorem are presented (see also [CheL]). Unfortunately this method cannot be applied if any minimal projection is not a co-minimal one. (Recall that a projection P_o is co-minimal if

$$\|Id - P_o\| = \operatorname{dist}(\mathrm{Id}, \mathscr{P}(X, Y)).)$$

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For example, this situation holds true for hyperplanes in $l_\infty^{(n)}$ (see e.g. [BlCh]). The second method is based on various Kolmogorov-type criteria. For more precise information about them the reader is referred to [BarL, BlCh, LE1, LE2, LE3, OdL]. Unfortunately, in many important cases the exact value of the constant $\lambda(Y, X)$ as well as a formula for minimal projection is unknown. In this paper we present a new formula for a minimal projection. We calculate the constant $\lambda(Y, l_\infty^{(4)})$ (R^4 with the maximum norm) for any two-dimensional subspace Y of $l_\infty^{(4)}$. We also determine a minimal projection in this case (Theorem 3.1) and present a complete characterization of its unicity.

Now let us introduce some notation. By $S(X)$ we denote the unit sphere in a normed space X and by $\mathrm{ext}(S(X))$ the set of its extremal points. The symbol $\mathscr{L}(X, Y)$ stands for the space of all linear, continuous mappings from X into Y. If Y is a linear subspace of X we write

$$\mathscr{L}_Y = \{L \in \mathscr{L}(X, Y) : L|_Y = 0\}. \tag{1.2}$$

It is easy to show that

$$\lambda(Y, X) = \mathrm{dist}(P, \mathscr{L}_Y) \tag{1.3}$$

for every $P \in \mathscr{P}(X, Y)$. If $X = l_\infty^{(n)}$ (R^n with the maximum norm) the symbol T_{ij}, $i, j \in \{1, ..., n\}$, stands for a transposition

$$T_{ij}(x_1, ..., x_i, ..., x_j, ..., x_n) = (x_1, ..., x_j, ..., x_i, ..., x_n), \tag{1.4}$$

where $x = (x_1, ..., x_n) \in l_\infty^{(n)}$.

Now we will present some notions and results which will be of use later. Let X be a normed space and let $x \in X$. Set

$$E(x) = \{f \in \mathrm{ext}(S(X^*)) : f(x) = \|x\|\} \tag{1.5}$$

Definition 1.1 [SW, Definition 5.1]. Let X be a real normed space, $x \in X \backslash \{0\}$, and let Y be an n-dimensional linear subspace of X. A set $I = \{f_1, ..., f_k\} \subset \mathrm{ext}(S(X^*))$ is called an I-set if there exist positive numbers $\lambda_1, ..., \lambda_k$ such that

$$\sum_{i=1}^{k} \lambda_i f_i|_Y = 0. \tag{1.6}$$

If moreover $I \subset E(x)$ then I is called an I-set with respect to x. An I-set I is said to be minimal if there is no proper subset of I which forms an I-set. A minimal I-set is called regular iff $k = n+1$ (by the Carathéodory theorem $n+1$ is the largest possible number).

The role of regular I-sets is illustrated by the next theorem.

THEOREM 1.2 [SW, Theorem 5.8]. *Let X be a real normed space and let $x \in X \setminus Y$, $y \in Y$. If there exists a regular I-set for $x - y$ then y is a strongly unique best approximation to x in Y. We recall that $y \in Y$ is a strongly unique best approximation to $x \in X$ if there is $r > 0$ depending only on x such that for any $w \in Y$*

$$\|x - w\| \geqslant \|x - y\| + r \cdot \|y - w\|.$$

From [RS] immediately follows the next theorem.

THEOREM 1.3 [RS]. *Let X be a finite-dimensional normed space. Then*

$$\operatorname{ext}(S(\mathscr{L}^*(X))) = \operatorname{ext}(S(X^*)) \otimes \operatorname{ext} S(X)), \tag{1.7}$$

where $(x^* \otimes x)(L) = x^*(Lx)$ for $x \in X$, $x^* \in X^*$, and $L \in \mathscr{L}(X)$.

LEMMA 1.4. (see e.g. [BlCh]). *Assume X is a normed space and let $Y \subset X$ be a subspace of codimension k, $Y = \bigcap_{i=1}^{k} \ker(f_i)$, where $f^i \in X^*$ are linearly independent. Let $P \in \mathscr{P}(X, Y)$. Then there exist $y^1, ..., y^k \in X$ satisfying $f^i(y^j) = \delta_{ij}$ for $i, j = 1, ..., k$ such that*

$$Px = x - \sum_{i=1}^{k} f^i(x)\, y^i \qquad \text{for} \quad x \in X. \tag{1.8}$$

On the other hand, if $y^1, ..., y^k \in Y$ satisfy $f^i(y_j) = \delta_{ij}$ then the operator $P = \mathrm{Id} - \sum_{i=1}^{k} f^i(\cdot)\, y^i$ belongs to $\mathscr{P}(X, Y)$.

LEMMA 1.5 (see e.g. [LE3, l. 2.4.4, p. 72]). *Let $X = l_\infty^{(n)}$ and let $Y =$ $\ker(f) \cap \ker(g)$, where $f, g \in S(X^*)$ are linearly independent. Let $P \in \mathscr{P}(X, Y)$, $P = \mathrm{Id} - f(\cdot)\, z - g(\cdot)\, w$, where $z, w \in X$. Then*

$$\|e_i \circ P\| = |1 - f_i z_i - g_i w_i| + \sum_{j \neq i} |f_j z_i + g_j w_i| \tag{1.9}$$

where $e_i(x) = x_i$ for $x \in X, i = 1, ..., n$. Moreover, $e_i(Px) = \|e_i \circ P\|$, for $x \in S(l_\infty^{(n)})$ iff

$$\operatorname{sgn}(f_j z_i + g_j w_i) = -\operatorname{sgn}(x_i) \qquad \text{for} \quad j \neq i$$

(if $f_j z_i + g_j w_i \neq 0$) and

$$\operatorname{sgn}(1 - f_i z_i - g_i w_i) = \operatorname{sgn}(x_i)$$

(if $1 - f_i z_i - g_i w_i \neq 0$).

LEMMA 1.6 (see e.g. [OdL, Prop. II.7.1, p. 82]). *Let Y_1, Y_2 be two linear subspaces of a normed space X. Suppose that there is a linear isometry T of X onto itself such that $T(Y_1)=Y_2$. Then $\lambda(Y_1, X)=\lambda(Y_2, X)$.*

DEFINITION 1.7. Let X be a normed space and let Y_1, Y_2 be two linear subspaces of X. It is said that Y_1 is equivalent to Y_2 if there is a linear isometry T of X onto itself such that $T(Y_1)=Y_2$.

2. TECHNICAL LEMMAS

LEMMA 2.1. *Let $Y\subset l_\infty^{(n)}$ be a subspace of codimension two, $Y=\ker(f)\cap\ker(g)$, where $f, g\in S(l_1^{(n)})$ are two linearly independent functionals. Then there is a linear subspace $Y_1\subset l_\infty^{(n)}$ equivalent to Y such that $Y_1=\ker(f^1)\cap\ker(g^1)$ where $f^1, g^1\in S(l_1^{(n)})$ are of the form $f^1=(f_1^1, 0, f_3^1, ..., f_n^1)$, $g^1=(0, g_2^1, ..., g_n^1)$, $f_1^1>0$, $g_2^1>0$, and $f_i^1, g_i^1\geqslant 0$ for $i=3, ..., n$.*

Proof. Since f, g are linearly independent,

$$\det(A_{ij})=\det\begin{pmatrix} f_i & g_i \\ f_j & g_j\end{pmatrix}\neq 0$$

for some $i\neq j$. Consider two systems of linear equations

$$A_{ij}u=(1,0), \qquad A_{ij}w=(0,1).$$

Let $u=(u_1, u_2)$ and $w=(w_1, w_2)$ be the solutions of these systems. Put $h=T_{1i}\circ T_{2j}(u_1 f+u_2 g)$, $l=T_{1i}\circ T_{2j}(w_1 f+w_2 g)$ (see (1.4)). We can assume without loss of generality that $h, l\in S(l_1^{(n)})$. Put $Z_1=\ker(h)\cap\ker(l)$. If $h, l\geqslant 0$, then h, l, Z_1 satisfy the requirements of the lemma. If not, then consider $k=|h|$, $m=(0, l_2, m_3, ..., m_n)$, where $m_i=|l_i|$ if $h_i=0$ and $m_i=l_i\operatorname{sgn}(h_i)$ in the opposite case. If $m\geqslant 0$ then k, m, and $Z_1=\ker(k)\cap\ker(m)$ satisfy the requirements of the lemma. If not, choose $i_o\geqslant 3$ such that

$$k_{i_o}/m_{i_o}=\max\{k_i/m_i\colon m_i<0\}.$$

Put

$$f^1=T_{1i_o}(k), \qquad u=T_{1i_o}(k-(k_{i_o}/m_{i_o})\cdot m)$$

and

$$g^1=(u_1, -u_2, u_3, ..., u_n).$$

It is clear that f^1, g^1, $Y_1 = \ker(f^1) \cap \ker(g^2)$ satisfy the requirements of the lemma (we can assume without loss of generality that $\|g\|_1 = 1$). The lemma is proved.

LEMMA 2.2. *Let $f, g \in S(l_1^{(4)})$ be of the form*

$$\begin{aligned} &f = (f_1, 0, f_3, f_4), && g = (0, g_2, g_3, g_4), \\ &f_i > 0, \quad \text{for} \quad i \neq 2 && g_i > 0 \quad \text{for} \quad i \neq 1. \end{aligned} \tag{2.1}$$

Put $Y = \ker(f) \cap \ker(g) \subset l_\infty^{(4)}$. Then $\|e_i|_Y\| = 1$ for $i = 1, 2, 3, 4$ if and only if

$$f_i \leqslant 1/2, \quad g_i \leqslant 1/2 \qquad \text{for} \quad i = 1, 2, 3, 4, \tag{2.2}$$

$$\left| \det \begin{pmatrix} f_1 & f_3 \\ g_2 & g_3 \end{pmatrix} \right| \leqslant \left| \det \begin{pmatrix} f_3 & f_4 \\ g_3 & g_4 \end{pmatrix} \right| \tag{2.3}$$

and

$$\left| \det \begin{pmatrix} f_1 & f_4 \\ g_2 & g_4 \end{pmatrix} \right| \leqslant \left| \det \begin{pmatrix} f_3 & f_4 \\ g_3 & g_4 \end{pmatrix} \right|. \tag{2.4}$$

Proof. Suppose that that (2.2) does not hold. We may assume without loss of generality that $f_i > 1/2$ for some $i \neq 2$. Then it is clear that $\|e_i|_{\ker(f)}\| < 1$ and consequently $\|e_i|_Y\| < 1$, a contradiction. If (2.3) or (2.4) are not satisfied, then consider $h = f + (-f_3/g_3) \cdot g$ and $l = f + (-f_4/g_4) \cdot g$. It is easy to see by definitions that $|h_i| > \sum_{j \neq i} |h_j|$ for $i = 1$ or $i = 2$ or $|l_i| > \sum_{j \neq i} |l_j|$ for $i = 1$ or $i = 2$. Reasoning as above we get a contradiction. Now assume that (2.2), (2.3), and (2.4) are satisfied. To prove that $\|e_i|_Y\| = 1$ for $i = 3, 4$ consider a system of linear equations

$$\begin{aligned} \lambda_1 f_1 + f_3 - f_4 &= 0, \\ \lambda_2 g_2 + g_3 - g_4 &= 0. \end{aligned}$$

By (2.2), $|\lambda_i| \leqslant 1$ for $i = 1, 2$ which gives the result. To prove that $\|e_i|_Y\| = 1$ for $i = 1, 2$ consider a system

$$\begin{aligned} -f_1 + \lambda_1 f_3 + \lambda_2 f_4 &= 0, \\ -g_2 + \lambda_1 g_3 + \lambda_2 g_4 &= 0. \end{aligned}$$

By Cramer's rule, (2.3), and (2.4), $|\lambda_i| \leqslant 1$. The lemma is proved.

COROLLARY 2.3. *Let Y be as in Lemma 2.2. If in* (2.2), (2.3), *or* (2.4) *we have equality then Y can be approximated* (*in the sense of the Banach–Mazur* *distance) by a sequence of two-dimensional subspaces* $Y_n \subset l_\infty^{(4)}$ *with* $\|e_i|_{Y_n}\| < 1$ *for some* $i \in \{1, 2, 3, 4\}$.

Proof. Suppose that we have equality in (2.2). we may assume without loss of generality that $f_i = 1/2$ for some $i = 1, 3, 4$. Take for $n \in N$, $f^n \in S(l_1^{(4)})$ with $f_i^n > 1/2$ such that $f^n \to f$ if $n \to \infty$. Put $Y_n = \ker(f^n) \cap \ker(g)$. It is clear that $\|e_i|_{Y_n}\| < 1$ and $d(Y, Y_n) \to 1$. (The symbol $d(Y, Y_n)$ denotes the Banach–Mazur distance between Y and Y_n). If we have equality in (2.3) or (2.4), the same reasoning applied to the functionals h, l (see the proof of Lemma 2.2) gives the result.

LEMMA 2.4. *Let* f, g *be as in Lemma* 2.2. *Suppose that* (2.2) *is satisfied. If* $f_3 > g_3$ ($< g_3$, *resp.*) *and* $f_4 > g_4$ ($< g_4$, *resp.*) *then* (2.3) *or* (2.4) *does not hold.*

Proof. We can assume without loss of generality that $f_3 > g_3$ and $f_4 > g_4$. Since $\|f\|_1 = \|g\|_1 = 1$, $g_2 > f_1$. Hence

$$\left|\det\begin{pmatrix} f_1 & f_i \\ g_2 & g_i \end{pmatrix}\right| = \det\begin{pmatrix} f_i & f_1 \\ g_i & g_2 \end{pmatrix}$$

for $i = 3, 4$. If

$$\det\begin{pmatrix} f_3 & f_4 \\ g_3 & g_4 \end{pmatrix} \geqslant 0$$

then

$$\det\begin{pmatrix} f_4 & f_1 \\ g_4 & g_2 \end{pmatrix} = f_4 - g_4 + \det\begin{pmatrix} f_3 & f_4 \\ g_3 & g_4 \end{pmatrix}$$
$$> \det\begin{pmatrix} f_3 & f_4 \\ g_3 & g_4 \end{pmatrix}$$

which contradicts (2.4). If

$$\det\begin{pmatrix} f_4 & f_3 \\ g_4 & g_3 \end{pmatrix} > 0,$$

by the same reasoning we infer that (2.3) is not satisfied.

LEMMA 2.5. *Suppose that* $f, g \in S(l_1^{(4)})$ *satisfy* (2.1), (2.2), (2.3), *and* (2.4) (*with the strict inequalities*). *Let* $f_3 > g_3$. *Define*

$$\phi_1^j = e_j \otimes x_1 = (1, 1, -1, -1) \qquad \textit{for} \quad j = 1, 2,$$
$$\phi_2 = e_3 \otimes x_3 = (-1, -1, 1, -1)$$

$$\phi_3 = e_3 \otimes y_3 = (-1, 1, 1, -1)$$
$$\phi_4 = e_4 \otimes x_4 = (-1, -1, -1, 1)$$
$$\phi_5 = e_4 \otimes y_4 = (1, -1, -1, 1).$$

Then $\{\phi_1^j, ..., \phi_5\}$ $(j=1, 2)$ *is a minimal regular I-set* (*see Definition* 1.1) *with respect to the* $\mathscr{L}_Y$ (*see* (1.2)).

Proof. Suppose that $j=1$. Consider the equation

$$\phi_1^1|_{\mathscr{L}_Y} + \sum_{i=2}^{5} \lambda_i \circ \phi_i|_{\mathscr{L}_Y} = 0 \tag{2.5}$$

with unknown variables λ_i, $i=2, 3, 4, 5$. Note that $\dim(\mathscr{L}_Y)=4$ and the mappings $M_1 = f(\cdot)\, w_1$, $M_2 = g(\cdot)\, w_1$, $M_3 = f(\cdot)\, w_2$, and $M_4 = g(\cdot)\, w_2$ form a basis of $\mathscr{L}_Y$. (Here $w_1 = (-f_3/f_1, -g_3/g_2, 1, 0)$, $w_2 = (-f_4/f_1, -g_4/g_2, 0, 1)$.) Hence (2.5) is equivalent to

$$\begin{aligned} \lambda_2 f(x_3) + \lambda_3 f(y_3) &= (f_3/f_1) \cdot f(x_1) \\ \lambda_2\, g(x_3) + \lambda_3\, g(y_3) &= (f_3/f_1) \cdot g(x_1) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \lambda_4 f(x_4) + \lambda_5 f(y_4) &= (f_4/f_1) \cdot f(x_1) \\ \lambda_4\, g(x_4) + \lambda_5\, g(y_4) &= (f_4/f_1) \cdot g(x_1) \end{aligned} \tag{2.7}$$

Applying the Cramer rule we easily get $\lambda_i = (f_3/f_1) \cdot A_i/C$ for $i=2, 3$ and $\lambda_i = (f_4/f_1) \cdot B_{i-1}/D$ for $i=4, 5$, where

$$A_2 = \det\begin{pmatrix} f(x_1) & f(y_3) \\ g(x_1) & g(y_3) \end{pmatrix} \qquad A_3 = \det\begin{pmatrix} f(x_3) & f(x_1) \\ g(x_3) & g(x_1) \end{pmatrix}$$

$$C = \det\begin{pmatrix} f(x_3) & f(y_3) \\ g(x_3) & g(y_3) \end{pmatrix} \qquad B_3 = \det\begin{pmatrix} f(x_1) & f(y_4) \\ g(x_1) & g(y_4) \end{pmatrix}$$

$$B_4 = \det\begin{pmatrix} f(x_4) & f(x_1) \\ g(x_4) & g(x_1) \end{pmatrix} \qquad D = \det\begin{pmatrix} f(x_4) & f(y_4) \\ g(x_4) & g(y_4) \end{pmatrix}.$$

To finish the proof it is sufficient to show that $C \neq 0$, $D \neq 0$, $\operatorname{sgn}(A_i) = \operatorname{sgn}(C)$ for $i=2, 3$ and $\operatorname{sgn}(B_{i-1}) = \operatorname{sgn}(D)$ for $i=4, 5$. By (2.1), (2.2), (2.3), (2.4), and elementary calculations we get

$$\begin{aligned} A_2 &= -[(1-2g_2)(1-2f_3) + (1-2f_1)(1-2g_4)] < 0, \\ C &= -(1-2f_3) \cdot 2g_2 < 0. \end{aligned}$$

Moreover, since $f_3 > g_3$, by Lemma 2.2 and 2.4

$$A_3 = \det\begin{pmatrix} 2f_3-1 & 2f_1-1 \\ 2g_3-1 & 2g_2-1 \end{pmatrix}$$
$$= 2\cdot\det\begin{pmatrix} f_1 & f_4 \\ g_2 & g_4 \end{pmatrix} - 2\cdot\det\begin{pmatrix} f_3 & f_4 \\ g_3 & g_4 \end{pmatrix} < 0.$$

Analogously,

$$B_3 = \det\begin{pmatrix} 2f_1-1 & 1-2f_3 \\ 2g_2-1 & 2g_4-1 \end{pmatrix} = (1-2f_1)(1-2g_4) + (1-2f_3)(1-2g_2) > 0,$$
$$D = \det\begin{pmatrix} 2f_4-1 & 1-2f_3 \\ 2g_4-1 & 2g_4-1 \end{pmatrix} = (1-2g_4)\,2f_1 > 0,$$
$$B_4 = \det\begin{pmatrix} 2f_4-1 & 2f_1-1 \\ 2g_4-1 & 2g_2-1 \end{pmatrix} = 2\cdot\det\begin{pmatrix} f_1 & f_3 \\ g_2 & g_3 \end{pmatrix} + 2\cdot\det\begin{pmatrix} f_3 & f_4 \\ g_3 & g_4 \end{pmatrix} > 0.$$

Consequently, $\lambda_i > 0$ for $i = 2, ..., 5$ which gives the result.

The same reasoning applies to ϕ_i^2, since $g_i/g_2 > 0$ for $i = 3, 4$. The lemma is proved.

LEMMA 2.6. *Let* $f, g \in S(l_1^{(4)})$ *be of the form* $f = (f_1, 0, f_3, f_4)$, $g = (0, g_2, g_3, g_4)$. *Assume that* f, g *satisfy* (2.1) *and*

$$f_3 g_4 - g_3 f_4 > 0. \tag{2.8}$$

Define

$$h^1 = ((f_3/g_3)\cdot g_4 - f_4, 0, (f_3/g_3)\cdot g_2, f_1),$$
$$l^1 = (0, f_3 - (f_4/g_4)\cdot g_3, (f_4/g_4)\cdot g_2, f_1),$$
$$h = h^1/\|h\|, \qquad l = l/\|l\|.$$

Then $h_3 l_4 - l_3 h_4 > 0$ *and*

$$f_3 > g_2 + g_3(1-2f_1),$$

if and only if

$$h_3 < l_2 + l_3(1-2h_1).$$

Proof. Note that

$$h_3 l_4 - l_3 h_4 = (\|h^1\|\cdot\|l^1\|)^{-1})\cdot f_1\, g_2(f_3/g_3 - f_4/g_4) > 0$$

by (2.1) and (2.8). Now suppose that $h_3 < l_2 + l_3(1-2h_1)$. This means that

$$h_3^1/\|h^1\| < (l_2^1\,\|h^1\| + l_3^1(\|h^1\| - 2h_1^1))/(\|h^1\|\cdot\|l^1\|)$$

and consequently

$$h_3^1\,\|l^1\| < (l_2^1 + l_3^1)\,\|h^1\| - 2l_3^1 h_1^1. \tag{2.9}$$

Now we rewrite (2.9) in terms of the coordinates of f and g and we show that the following 10 inequalities are equivalent:

$$\begin{aligned}&(f_3/g_3)\cdot(g_2/g_4)(f_3g_4 - f_4g_3 + f_4g_2 + f_1g_4)\\ &\quad < [(f_3g_4 - f_4g_3 + f_4g_2)(f_3g_4 - f_4g_3 + f_3g_2 + f_1g_3)\\ &\qquad - 2f_4g_2(f_3g_4 - g_3f_4)]/(g_4g_3)\end{aligned}$$

$$\begin{aligned}&f_3g_2(f_3g_4 - f_4g_3) + f_3g_2(f_4g_2 + f_1g_4)\\ &\quad < (f_3g_4 - f_4g_3)(f_3g_4 - f_4g_3 + f_3g_2 + f_1g_3 - f_4g_2)\\ &\qquad + f_4g_2(f_3g_2 + f_1g_3)\end{aligned}$$

$$\begin{aligned}&f_3g_2(f_4g_2 + f_1g_4)\\ &\quad < f_4g_2(f_3g_2 + f_1g_3)\\ &\qquad + (f_3g_4 - f_4g_3)(f_3g_4 - f_4g_3 - f_4g_2 + f_1g_3)\end{aligned}$$

$$\begin{aligned}&f_3g_2f_1g_4\\ &\quad < f_4g_2f_1g_3\\ &\qquad + (f_3g_4 - f_4g_3)(f_3g_4 - f_4g_3 - f_4g_2 + f_1g_3)\end{aligned}$$

(by (2.8)) if and only if

$$\begin{aligned}g_2f_1 &< f_3g_4 - f_4g_3 + f_1g_3 - f_4g_2\\ 0 &< f_3g_4 - g_2(f_4 + f_1) + g_3(f_1 - f_4)\\ 0 &< f_3g_4 - g_2(1 - f_3) + g_3(f_1 - f_4)\\ 0 &< -g_2 + f_3(1 - g_3) + g_3(f_1 - f_4)\\ f_3 &> g_2 + g_3(f_3 + f_4 - f_1)\\ f_3 &> g_2 + g_3(1 - 2f_1).\end{aligned}$$

The lemma is proved.

LEMMA 2.7. *Let $f, g \in S(l_1^{(4)})$ be as in Lemma 2.6. Put $h=(f_4, 0, f_3, f_1)$, $l^1=(0, (f_4/g_4)\cdot g_2, f_3-(f_4/g_4)\cdot g_3, f_1)$, $l=l^1/\|l^1\|$. Then the following two conditions are equivalent*:

$$f_3 \leqslant g_2+g_3(1-2f_1) \qquad \text{and} \qquad g_4>f_1+f_4(1-2g_2) \tag{2.10}$$

$$h_3 \leqslant l_2+l_3(1-2h_1) \qquad \text{and} \qquad l_4<h_1+h_4(1-2h_2). \tag{2.11}$$

Moreover, $h_3 l_4 - h_4 l_3 > 0$.

Proof. Note that the following seven conditions are equivalent:

$$h_3 \leqslant l_2+l_3(1-2h_1)$$

$$h_3\cdot\|l^1\| \leqslant l_2^1+l_3^1(1-2h_1)$$

$$(f_2/g_4)\cdot(f_4 g_2+f_3 g_4-f_4 g_3+f_1 g_4) \leqslant [f_4 g_2(f_3 g_4-f_4 g_3)(1-2f_4)]/g_4$$

$$f_3 g_2 \leqslant g_2-f_1 g_3-f_3 g_4+f_4 g_3$$

$$f_3(g_2+g_4) \leqslant g_2+g_3(f_4-f_1)$$

$$f_3(g_2+g_3+g_4) \leqslant g_2+g_3(f_3+f_4-f_1)$$

$$f_3 \leqslant g_2+g_3(1-2f_1).$$

Analogously, by elementary calculations, the following eight conditions are equivalent:

$$l_4<h_1+h_4(1-2l_2)$$

$$l_4^1<h_1\cdot\|l^1\|+h_4(\|l^1\|-2l_2^1)$$

$$f_1 g_4<(f_1+f_4)(f_4 g_2+f_3 g_4-f_4 g_3+f_1 g_4)-2f_4 g_2 f_1$$

$$f_1 g_4+f_1 f_4 g_2<f_1 g_4(f_3+f_1+f_4)+f_4(f_4 g_2+f_3 g_4-f_4 g_3-f_1 g_3)$$

$$f_1 g_2<f_4 g_2+f_3 g_4-f_4 g_3-f_1 g_3$$

$$f_3 g_4>f_4(g_3-g_2)+f_1(g_2+g_3)$$

$$g_4(f_1+f_3+f_4)>f_1(g_2+g_3+g_4)+f_4(g_4+g_3-g_2)$$

$$g_4>f_1+f_4(1-2g_2).$$

This proves the equivalence of (2.10) to (2.11).

Note that

$$h_3 l_4-h_4 l_3=\|l^1\|^{-1}\cdot(f_3 f_1-f_1(f_3-(f_4/g_4)\cdot g_3))$$
$$=f_1 f_4 g_3/(g_4\cdot\|l^1\|)>0.$$

The lemma is proved.

3. THE MAIN RESULTS

Now we can prove the main result of this paper.

THEOREM 3.1. *Let $f, g \in Sl_1^{(4)}$ satisfy* (2.1), (2.2), (2.3), *and* (2.4) *with strict inequalities. Assume additionally that $f_3 > g_3$ and*

$$\begin{aligned} f_3 &\leqslant g_2 + g_3(1-2f_1), \\ g_4 &\leqslant f_1 + f_4(1-2g_2). \end{aligned} \tag{3.1}$$

Put $Y = \ker(f) \cap \ker(g)$. Then $\lambda(Y, l_\infty^{(4)}) = \max\{a, b\}$, where

$$\begin{aligned} a = 1 + (g_2/(1-2g_2) + g_3(1-2f_1)/[(1-2g_2)(1-2f_3)] \\ + g_4/(1-2g_4))^{-1} \end{aligned} \tag{3.2}$$

$$\begin{aligned} b = 1 + (f_1/(1-2f_1) + f_4(1-2g_2)/[(1-2f_1)(1-2g_4)] \\ + f_3/(1-2f_3))^{-1} \end{aligned} \tag{3.3}$$

Moreover, there is a strongly unique (*in particular, a unique*) *minimal projection. This projection is determined by the vectors z, $w \in R^4$, $f(z) = g(w) = 1$, $f(w) = g(z) = 0$* (*compare with Lemma* 1.4) *of the form*:

$$\begin{aligned} &z_4 = 0, \quad z_3 = (a-1)/(1-2f_3), \quad z_2 = -(g_3/g_2)\, z_3, \quad z_1 = (1-f_3 z_3)/f_1; \\ &w_4 = (a-1)/(1-2g_4), \quad w_3 = 0, \quad w_2 = (a-1-(1-2f_1)\, z_2)/(1-2g_2), \\ &w_1 = -(f_4/f_1) w_4) \end{aligned} \tag{3.4}$$

if $a \geqslant b$ or

$$\begin{aligned} &w_4 = (b-1)/(1-2g_4), \quad w_3 = 0, \quad w_2 = (1-g_4 w_4)/g_2, \quad w_1 = -(f_4/f_1) w_4 \\ &z_4 = 0, \quad z_3 = (b-1)(1-2f_3), \quad z_2 = -(g_3/g_2)\, z_3, \\ &z_1 = (b-1-(1-2g_2) w_1)/(1-2f_1) \end{aligned} \tag{3.5}$$

if $b > a$.

Proof. Suppose $a \geqslant b$ and consider a system of equations

$$\begin{aligned} &\phi_1^2(Id - f(\cdot)\, z - g(\cdot) w) = d_g \\ &\phi_i(Id - f(\cdot)\, z - g(\cdot) w) = d_g \qquad \text{for} \quad i = 2, 3, 4, 5; \end{aligned} \tag{3.6}$$

$$f(z)=g(w)=1, \qquad f(w)=g(z)=0,$$

with unknown variables d_g, $w=(w_1, ..., w_4)$, $z=(z_1, ..., z_4)$, where ϕ_1^2, $\phi_i (i=2, ..., 5)$ are as in Lemma 2.5. By the definition of ϕ_1^2 and ϕ_i (see Theorem 1.3) (3.6) can be rewritten in the form:

$$\begin{aligned} d_g-1&=(1-2f_1)\,z_2+(1-2g_2)w_2, \\ d_g-1&=(1-2f_3)\,z_3+(1-2g_3)w_3 \\ d_g-1&=(1-2f_3)\,z_3+(2g_4-1)w_3 \\ d_g-1&=(1-2f_4)\,z_4+(1-2g_4)w_4 \\ d_g-1&=(2f_3-1)\,z_4+(1-2g_4)w_4 \\ f(z)=g(w)&=1, \qquad f(w)=g(z)=0. \end{aligned} \tag{3.7}$$

From the second and third equation of (3.7) we get

$$w_3=0, \qquad z_3=(d_g-1)/(1-2f_3).$$

Analogously from the fourth and fifth equation

$$z_4=0, w_4=(d_g-1)/(1-2g_4).$$

From the first equation we have $w_2=(d_g-1-(1-2f_1)\,z_2)/(1-2g_2)$. Since $g(z)=0$, $z_2=(-g_3/g_2)\,z_3$. Applying the formulas for z_3, z_2, w_2, w_4 to the equation $g(w)=1$ we easily get that $d_g=a$, where a is given by (3.2). Since $f(w)=0$ and $f(z)=1$ we obtain $w_1=(-f_1/f_4)w_4$, $z_1=(1-f_3z_3)/f_1$.

Put $P_o=Id-f(\cdot)\,z-g(\cdot)w$, where z, w are the solution of (3.7). By Lemma 1.4, $P_o \in \mathscr{P}(l_{\infty}^{(4)}, Y)$. Now suppose that we have proved

$$a=d_g=\phi_1^2(P_o)=\|P_o\| \tag{3.8}$$

and

$$a=d_g=\phi_i(P_o)=\|P_o\| \qquad \text{for} \quad i=2, ..., 5. \tag{3.9}$$

By Lemma 2.5 the functionals $\phi_1^2, \phi_i, i=2, ..., 5$, form a minimal regular I-set with respect to $\mathscr{L}_Y$ (see (1.2)). By Theorem 1.3, $\phi_1^1, \phi_i \in \text{ext}(S(l_{\infty}^{(4)})$ for $i=2, ..., 5$. From (3.8) and (3.9) it follows that this I-set is contained in $E(P_o)$ (see (1.5)). By Theorem 1.2, 0 is a strongly unique best approximation for P_o in $\mathscr{L}_Y$, which means that P_o is a unique minimal projection (see (1.3)) in the case $a \geqslant b$ (see (3.2), (3.3)). To prove (3.8) and (3.9) we show that

$$\|e_i \circ P_o\|=d_g=a \qquad \text{for} \quad i=2, 3, 4$$

and

$$\|e_1 \circ P_o\| \leqslant a.$$

By Lemma 1.5,

$$\|e_i \circ P_o\| = |1 - f_i z_i - g_i w_i| + \sum_{j \neq i} |f_j z_i + g_j w_i|$$

for $i = 1, 2, 3, 4$. By the definition of ϕ_1^2 and ϕ_j for $j = 2, ..., 5$ it is necessary to show that

$$\|e_i \circ P_o\| = e_i(P_o x^i) = e_i(P_o y^i), \qquad \text{for} \quad i = 3, 4, \tag{3.10}$$

$$\|e_2 \circ P_o\| = e_2(P_o x^2). \tag{3.11}$$

and

$$\|e_1 \circ P_o\| = e_1(P_o x^2) \leqslant a. \tag{3.12}$$

To prove (3.10), first note that $f_1 z_4 + g_1 w_4 = 0$, since $z_4 = 0$ and $g_1 = 0$. Hence if $e_4(P_o z) = \|P_o\|$ then z_1 can be arbitrary. By Lemma 1.5 it is necessary to show that $\operatorname{sgn}(1 - g_4 w_4) = x_4 = y_4 = 1$, $\operatorname{sgn}(g_j w_4) = -x_j = -y_j = 1$ for $j = 2, 3$. But this is true, since $w_4, g_2, g_3 > 0$ (by (3.2), $d_g = a > 1$). Since $w_3 = 0$, $z_3 > 0$, and $f_2 = 0$ the same reasoning applies for $i = 3$. Consequently, (3.10) is proved. To show (3.11) we verify that $f_j z_2 + g_j w_2 \geqslant 0$ for $j = 3, 4$, $1 - g_2 w_2 > 0$, and $f_1 z_2 < 0$. Since $z_2 = -(g_3/g_2)\, z_3$, $z_3 > 0$ and $w_4 > 0$ the last two inequalities hold true. Note that by (2.4) $f_3 g_4 - f_4 g_3 > f_1 g_4 - g_2 f_4$ which is equivalent to $(g_4/f_4)(1 - 2f_1) > 1 - 2g_2$. Since $z_2 < 0$, by the first equation from (3.7), $w_2 > 0$. Consequently,

$$\begin{aligned}(1 - 2f_1)\, z_2 &+ (g_4/f_4)(1 - 2f_1) w_2 \\ &> (1 - 2f_1)\, z_2 + (1 - 2g_2) w_2 = d_g - 1 = a - 1 > 0.\end{aligned}$$

Hence $f_4 z_2 + g_4 w_2 > 0$, as desired.

Now note that

$$\begin{aligned} f_3 z_2 + g_3 w_2 &= (g_3/g_2)(-f_3)\, z_3 + g_3(1 - g_4 w_4)/g_2 \\ &= (g_3/g_2) \cdot ((-f_3)(d_g - 1)/(1 - 2f_3) + 1 - g_4(d_g - 1)/(1 - 2g_4)). \end{aligned}$$

Hence

$$f_3 z_2 + g_3 w_2 \geqslant 0$$

if and only if

$$(d_g-1)(f_3/(1-2f_3)+g_4/(1-2g_4)\leqslant 1. \tag{3.13}$$

Since $d_g=a$, by (3.2), the last inequality is equivalent to

$$f_3\leqslant [g_2(1-2f_3)+g_3(1-2f_1)]/(1-2g_2)$$

which is the same as the first inequality in (3.1). Hence (3.11) is proved.

To show (3.12), first we verify that $z_1>0$. To do this, note that $f_1z_1=1-f_3z_3$. Hence it is necessary to show that $1-f_3z_3>0$. Since $z_3=(d_g-1)/(1-2f_3)$ this is equivalent to $(d_g-1)^{-1}>f_3/(1-2f_3)$ which immediately follows from (3.13). Moreover, $w_1=(-f_4/f_1)\,w_4<0$ since $w_4=(a-1)/(1-2g_4)>0$. Consequently $\operatorname{sgn}(1-f_1z_1-g_1w_1)=1$ and $\operatorname{sgn}(f_2z_1+g_2w_1)=-1$. Now we show that $f_3z_1+g_3w_1>0$ and $f_4z_1+g_4w_1\geqslant 0$. First note that the following four inequalities are equivalent:

$$\begin{gathered} f_3z_1+g_3w_1>0\\ (f_3/f_1)(1-f_3z_3)+(g_3/f_1)((-f_4)\,w_4)>0\\ f_3-(d_g-1)(f_3)^2/(1-2f_3)-(d_g-1)\,f_4\,g_3/(1-2g_4)>0\\ (d_g-1)^{-1}>f_3/(1-2f_3)+f_4\,g_3/(f_3(1-2g_4)). \end{gathered} \tag{3.14}$$

Since $f_3>g_3$, by (2.3), (3.1), $f_4\,g_3/f_3<g_4$. Hence (3.14) follows immediately from (3.13) and consequently $f_3z_1+g_3w_1>0$. Now note that the following three conditions are equivalent:

$$\begin{gathered} f_4z_1+g_4w_1\geqslant 0\\ (f_4/f_1)(1-f_3z_3)+(f_4/f_1)(-g_4w_4)\geqslant 0\\ 1-f_3(d_g-1)/(1-2f_3)-g_4(d_g-1)/(1-2g_4)\geqslant 0. \end{gathered}$$

But the last inequality is equivalent to (3.13) which is true. Consequently, we have proved that $\|e_1\circ P_o\|=e_1(P_ox_2)$. Hence

$$\|e_1\circ P_o\|=(1-2g_2)w_1+(1-2f_1)\,z_1+1.$$

To end the proof of (3.12) note that the following five inequalities are equivalent:

$$\|e_1 \circ P_o\| \leqslant a = d_g$$

$$(1-2g_2)w_1 + (1-2f_1)z_1 + 1 \leqslant d_g$$

$$-(1-2g_2)\ f_4(d_g-1)/(f_1)(1-2g_4) \\ +(1-2f_1)(1-f_3(d_g-1)/(1-2f_3))/f_1 + 1 \leqslant d_g$$

$$(d_g-1)(-1-(1-2g_2)\ f_4/(f_1(1-2g_4)) - (1-2f_1)\ f_3/(f_1(1-2f_3)) \\ \leqslant (2f_1-1)/f_1$$

$$(d_g-1)^{-1} \\ \leqslant f_1/(1-2f_1) + (1-2g_2)\ f_4/((1-2g_4)(1-2f_1)) + f_3/(1-2f_3).$$

By (3.3) the last inequality is equivalent to $(a-1)^{-1} \leqslant (b-1)^{-1}$ and consequently to $a \geqslant b$. So the theorem is proved in the case $a \geqslant b$. If $b > a$ let us consider $Y_1 = \ker(f^1) \cap \ker(g^1)$, where $f^1 = (g_2, 0, g_4, g_3)$ and $g^1 = (0, f_1, f_4, f_3)$. Note that Y_1 is equivalent (see Definition 1.7) to Y. Repeating the same argument for Y_1 we prove the case $b > a$ for Y.

The proof of Theorem 3.1 is complete.

Remark 3.2. Note that if Y satisfies all the assumptions of Theorem 3.1 apart from (3.1) then by Lemmas 2.6 and 2.7 we can replace Y by an equivalent (see Definition 1.7) subspace $Y_1 = \ker(h) \cap \ker(l)$, where $h, l \in S(l_1^{(4)})$ are as in Lemma 2.6 or 2.7. In this case (3.1) is satisfied. Moreover, (2.1), (2.2), (2.3), and (2.4) hold true for h, l with the strict inequalities (these conditions are invariant under linear isometry). Consequently, applying Theorem 3.1 to Y_1, we get the result for Y. If $f_3 \leqslant g_3$ then from (2.3), (2.4), and Lemma 2.4 it follows that $f_4 > g_4$. To apply Theorem 3.1 in this case we should consider $h = (f_1, 0, f_4, f_3)$, $l = (0, g_2, g_4, g_3)$. If (2.2), (2.3), or (2.4) is not satisfied then to compute $\lambda(Y, l_\infty^{(4)})$ we can apply Theorem 2.5 from [LE2] (see also [LE3, Theorem 2.4.6, p. 73]). For the convenience of the reader it will be now presented in a simpler form.

THEOREM 3.3 [LE2, Theorem 2.5]. *Suppose* $Y = \ker(f) \cap \ker(g)$ *where* $f, g \in S(l_1^{(n)})$ *satisfy* $f_1 \geqslant 1/2$, $g_2 > 0$, $f_2 = g_1 = 0$, *and* $g_i \geqslant 0$ *for* $i \geqslant 3$. *If* $g_2 \geqslant 1/2$ *then* $\lambda(Y, l_\infty^{(n)}) = 1$. *Moreover, the vectors* $z = (1/f_1, 0, ..., 0)$, $w = (0, 1/g_2, ..., 0)$ *determine a minimal projection* (*see Lemma* 1.4).

If $g_i < 1/2$ *for* $i = 1, ..., n$ *then*

$$\lambda(Y, l_\infty^{(n)}) = 1 + \left(\sum_{i=1}^{n} g_i/(1-2g_i)\right)^{-1}.$$

Moreover, the vectors $z = (1/f_1, 0, ..., 0)$ *and* $w = (w_1, ..., w_n)$ *where* $w_i = (\lambda(Y, l_\infty^{(n)}) - 1)/(1-2g_i)$ *for* $i \geqslant 2$ *and* $w_1 = -(\sum_{i \neq 1} f_i w_i)/f_1$ *determine a minimal projection in this case.*

Remark 3.4. By Lemmas 1.6 and 2.1, Theorems 3.1 and 3.3 permit us to calculate $\lambda(Y, l_\infty^{(4)})$ for any two-dimensional subspace Y of $l_\infty^{(4)}$. A complete characterization of the unicity of minimal projection in the case considered in Theorem 3.3 can be found in [LE2, Th. 3.1, 3.3, 3.4] or in [LE3, Th. 2.5.1, 2.5.3, 2.5.4, pp. 75–78].

EXAMPLE 3.5. Take $f=(2/5, 0, 2/5, 1/5)$, $g=(0, 2/5, 1/5, 2/5)$, and let $Y=\ker(f)\cap\ker(g)$. Then we have

$$\left|\det\begin{pmatrix} f_1 & f_3 \\ g_2 & g_3 \end{pmatrix}\right| = 2/25;$$

$$\left|\det\begin{pmatrix} f_1 & f_4 \\ g_2 & g_4 \end{pmatrix}\right| = 2/25;$$

$$\left|\det\begin{pmatrix} f_3 & f_4 \\ g_3 & g_4 \end{pmatrix}\right| = 3/25.$$

Hence it is easy to see that (2.1), (2.2), (2.3), and (2.4) are satisfied. Since

$$g_2 + g_3(1-2f_1) = f_1 + f_4(1-2g_2) = 2/5 + 1/25 > 2/5 = f_3 = g_4,$$

(3.1) is satisfied too. Consequently, by Theorem 3.1, $\lambda(Y, l_\infty^{(4)}) = 1 + 1/5$.

EXAMPLE 3.6. Let $f=(1/4+3c, 0, 1/2-5c, 1/4+2c)$, $g=(0, 2/5, 1/5, 2/5)$, and $Y=\ker(f)\cap\ker(g)$. It is easy to verify as in the previous example that (2.1), (2.2), (2.3), and (2.4) are satisfied for sufficiently small $c>0$. But the second inequality in (3.1) does not hold. Then applying Lemma 2.7 we can consider an equivalent subspace $Y_1=\ker(h)\cap\ker(l)$. Here h, l are as in Lemma 2.7. Hence the assumptions of Theorem 3.1 are satisfied. In our situation

$$h=(1/4+2c, 0, 1/2-5c, 1/4+3c),\ l=(0, 2+16c, 3-48c, 2+24c)/(7-8c).$$

Hence by Theorem 3.1

$$\lambda(Y, l_\infty^{(4)}) = 1 + \max\{(2+16c)/(3-40c) + (3-48c)(1-8c)/(3-40c)20c$$
$$+ (2+24c)/(3-56c))^{-1},$$
$$(1+8c)/(2-16c) + (1+12c)(3-40c)/(2-16c)(3-56c) + (1-10c)/20c)^{-1}\}.$$

EXAMPLE 3.7. Let $f=(9/32, 0, 7/16, 9/32)$, $g=(0, 9/32, 9/32, 7/16)$. It is also easy to verify that in this case (2.1), (2.2), (2.3), and (2.4) are satisfied. However, in this case, both inequalities in (3.1) do not hold true. Then by applying Lemma 2.6 we can consider an equivalent subspace $Y_1 = \ker(h) \cap \ker(l)$. Here h, l are as in Lemma 2.6. Hence the assumptions of Theorem 3.1 are satisfied. In our situation

$$h = (115/322, 0, 9/23, 81/322), \qquad l = (0, 115/322, 81/322, 9/23).$$

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